Explaining the method in RND_statistics_example.xls

Due to homogenity one can switch to the normalized situation with future = 1, time = 1 and rates = 0.

If pdf is the risk neutral density in log space (i.e. over log(strike), then $\mathrm{rnd} := \kappa \to \frac{\mathrm{pdf}(\ln(\kappa))}{\kappa}$ is the risk neutral density over the strikes.

Changing variables $\mu = \ln(\kappa)$ transforms the un-centered moments $\int_{-\infty}^{\infty} \mu^n \,\mathrm{pdf}(\mu)\, d\mu$ to $\int_0^{\infty} \frac{\ln(\kappa)^n \,\mathrm{pdf}(\ln(\kappa))}{\kappa}\, d\kappa$ which is $\int_0^{\infty} \ln(\kappa)^n \,\mathrm{rnd}(\kappa)\, d\kappa$ (so for example the usual mean becomes a logarithmic contract).

If a payoff is twice continuos differentiable (and has some regularity at the boundaries 0 and ∞) then its expectation value $\int_{-\infty}^{\infty} g(\kappa)\,\mathrm{rnd}(\kappa)\, d\kappa$ can be written as $g(F) + \int_0^{F} P(\kappa)\left(\frac{d^2}{d\kappa^2} g(\kappa)\right) d\kappa + \int_F^{\infty} C(\kappa)\left(\frac{d^2}{d\kappa^2} g(\kappa)\right) d\kappa$ with F = future and $P(\kappa)$, $C(\kappa)$ calls and puts w.r.t. the rnd.

For $K0 \le F$ this equals $g(F) + \int_0^{K0} P(\kappa)\left(\frac{d^2}{d\kappa^2} g(\kappa)\right) d\kappa + \int_{K0}^{\infty} C(\kappa)\left(\frac{d^2}{d\kappa^2} g(\kappa)\right) d\kappa + \int_{K0}^{F} (\kappa - F)\left(\frac{d^2}{d\kappa^2} g(\kappa)\right) d\kappa$

The last integral has an explicit solution (no Taylor series like in the VIX construction is needed) and one gets

$$\mathrm{Expectation}(g) = g(K0) - (K0 - 1)\, D(g)(K0) + \int_0^{K0} P(\kappa)\left(\frac{d^2}{d\kappa^2} g(\kappa)\right) d\kappa + \int_{K0}^{\infty} C(\kappa)\left(\frac{d^2}{d\kappa^2} g(\kappa)\right) d\kappa$$

One could try to approximate the integrals as one sum using linear interpolation (like in VIX), but they are peaked at K0, so they should be done separately. Since they are over (weighted) prices, and prices can be approximated quite reasonable by Taylor series up to 3rd order, one can use a scheme of interpolating prices by parabolas over the given strikes. This is the Simpson method - and it can be done for unequally spaced grids. The grid should have an unpair number of points (or an even number of subintervals) - but if a last one remains, one can interpolate from points before and write down the integral for the last strip.