

Maple 12.02, IBM INTEL NT, Dec 10 2008 Build ID 377066

Digits := 18

theDLL := "D:_Work\vc2005\hyp2f1_various\hyp2f1\release\hyp2f1.dll"

Define the functions MPL_2F1 (which is Maple's version) and hyp2f1_DLL (which is calling the DLL).

For comparing results one may wish to convert numerical inputs (decimals in Maple) to their nearest IEEE equivalent to have the same input. After that pre-processing a test routine can be feed, which gives both of the results and the relative and absolute errors.

prepare for testing

Functions

```
MPL_2F1 := proc(a, b, c, z) evalf(hypergeom([a, b], [c], z)) end proc
hyp2f1_mpl := proc(inputAarry::rtable(datatype = float[8]), result::rtable(datatype = float[8]))
option call_external, define_external(hyp2f1_mpl, C, inputAarry::ARRAY(1..8, float[8]),
result::ARRAY(1..2, datatype = float[8], NO_COPY), LIB = "D:\_Work\vc2005\hyp2f1_various\hyp2f1\release\hyp2f1.dll");
call_external(Array(1..10, [...], datatype = integer[4], readonly), false, args)
end proc
hyp2f1_DLL := proc(a, b, c, z)
local A, R, result;
A := Array(1..8, 0, order = Fortran_order, datatype = float[8]);
R := Array(1..2, 0, order = Fortran_order, datatype = float[8]);
R[1], R[2] := 0., 0.;
A[1] := ℞(evalf(a));
A[2] := ℑ(evalf(a));
A[3] := ℞(evalf(b));
A[4] := ℑ(evalf(b));
A[5] := ℞(evalf(c));
A[6] := ℑ(evalf(c));
A[7] := ℞(evalf(z));
A[8] := ℑ(evalf(z));
evalf((proc(AA, RR) hyp2f1_mpl(AA, RR) end proc)(A, R));
result := copy(R[1] + R[2]*I);
simplify(% , zero)
end proc
test_hyp2f1_DLL := proc(a, b, c, z)
local m, d;
m := evalf(MPL_2F1(a, b, c, z));
d := evalf(hyp2f1_DLL(a, b, c, z));
print('MPL' = m);
print('DLL' = d);
`error abs, rel` = m - d, abs((m - d) / m)
end proc
```

nearest IEEE rational

Naja, seine mantissa liegt zwischen 1 und 2, also anpassen:

```
pow := ^
nearest := proc(z) local x, y; Digits := 54; my_nearest_rational(℞(evalf(z))) + my_nearest_rational(ℑ(evalf(z)))*I end proc
```

pre-process data: new, regarding IEEE

```
preproc := proc(assignmentList)
local tstData, TstData, i;
Digits := 18; tstData := map('t → lhs(t) = nearest(rhs(t))', assignmentList); TstData := map(rhs, tstData); TstData := op(TstData)
end proc
```

test routine

```
testRoutine := proc(a, b, c, z)
local g, h;
gc( );
forget(hyp2f1_DLL);
```

```

g := evalf(hyp2f1_DLL(a, b, c, z));
Digits := Digits + 3;
h := MPL_2F1(a, b, c, z);
Digits := Digits - 3;
g := eval(nearest(g));
g := evalf(g);
h := eval(nearest(h));
h := evalf(h);
print('DLL' = g);
print('MPL' = h);
print('error' [ absolute, relative ] = abs(h - g), abs(1 - g / h));
if 4 < nargs then
    print('check with doubled digits:'); Digits := 2*Digits; h := MPL_2F1(TstData); print('MPL' = evalf[ Digits / 2 ](h))
end if
end proc

```

First: look at the 'usual critical case': points on the circle line, which are also on the diagonal (fix points at the usual linear transforms):

$$\begin{aligned}
 \text{tstData} &:= \left[a = \frac{1}{2}, b = \frac{1}{3}, c = \frac{1}{6}, z = e^{(1/3)I\pi} \right] \\
 \text{TstData} &:= \frac{1}{2} \text{pow}(2, 0), \frac{6004799503160661}{9007199254740992} \text{pow}(2, -1), \frac{6004799503160661}{9007199254740992} \text{pow}(2, -2), \frac{1}{2} \text{pow}(2, 0) + \frac{3900231685776981}{4503599627370496} I \text{pow}(2, 0) \\
 \text{DLL} &= 0.623714907712016986 + 0.958923198106978791 I \\
 \text{MPL} &= 0.623714907712016986 + 0.958923198106978902 I \\
 \text{error}_{\text{absolute, relative}} &= 0.111 \cdot 10^{-15}, 0.967481594953503273 \cdot 10^{-16}
 \end{aligned}$$

tests

1

$$\begin{aligned}
 \text{tstData} &:= \left[a = -3\pi, b = \sqrt{2}, c = e, z = \frac{1}{e^\pi} + 1 - 0.1 \cdot 10^{-289} I \right] \\
 \text{TstData} &:= \frac{2652839157010665}{4503599627370496} \text{pow}(2, 4), \frac{6369051672525773}{9007199254740992} \text{pow}(2, 1), \frac{6121026514868073}{9007199254740992} \text{pow}(2, 2), \\
 &\frac{587277226695055}{1125899906842624} \text{pow}(2, 1) - \frac{3511119404027961}{4503599627370496} I \text{pow}(2, -963) \\
 \text{DLL} &= 0.0560182538639225416 - 0.172417988082929226 \cdot 10^{-15} I \\
 \text{MPL} &= 0.0560182538639227706 - 0.172417988082930188 \cdot 10^{-15} I \\
 \text{error}_{\text{absolute, relative}} &= 0.2290 \cdot 10^{-15}, 0.4088 \cdot 10^{-14}
 \end{aligned}$$

2

$$\begin{aligned}
 \text{tstData} &:= [a = 1, b = 1, c = 2.2, z = 1.9] \\
 \text{TstData} &:= \frac{1}{2} \text{pow}(2, 1), \frac{1}{2} \text{pow}(2, 1), \frac{2476979795053773}{4503599627370496} \text{pow}(2, 2), \frac{4278419646001971}{4503599627370496} \text{pow}(2, 1) \\
 \text{DLL} &= 0.315767495606526449 - 1.70873729899197846 I \\
 \text{MPL} &= 0.315767495606590731 - 1.70873729899200000 I \\
 \text{error}_{\text{absolute, relative}} &= 0.677948901024258612 \cdot 10^{-13}, 0.390148986414175297 \cdot 10^{-13}
 \end{aligned}$$

3

$$\begin{aligned}
 \text{tstData} &:= [a = 1.756, b = 4.76, c = 2.20, z = 0.85] \\
 \text{TstData} &:= \frac{7908320945662591}{9007199254740992} \text{pow}(2, 1), \frac{2679641778285445}{4503599627370496} \text{pow}(2, 3), \frac{2476979795053773}{4503599627370496} \text{pow}(2, 2), \frac{7656119366529843}{9007199254740992} \text{pow}(2, 0) \\
 \text{DLL} &= 2443.22685352899316 \\
 \text{MPL} &= 2443.22685352899134 \\
 \text{error}_{\text{absolute, relative}} &= 0.182 \cdot 10^{-11}, 0.74 \cdot 10^{-15}
 \end{aligned}$$

4

$$\begin{aligned}
 \text{tstData} &:= [a = 1.756, b = 4.76, c = 2.20, z = 1.95 I] \\
 \text{TstData} &:=
 \end{aligned}$$

$$\frac{7908320945662591}{9007199254740992} \text{pow}(2, 1), \frac{2679641778285445}{4503599627370496} \text{pow}(2, 3), \frac{2476979795053773}{4503599627370496} \text{pow}(2, 2), \frac{8782019273372467}{9007199254740992} \text{pow}(2, 1)$$

$$\text{DLL} = -0.0291190062043904692 - 0.0104896762749490478 \text{ I}$$

$$\text{MPL} = -0.0291190062043906114 - 0.0104896762749490859 \text{ I}$$

$$\text{error}_{\text{absolute, relative}} = 0.147215658134588387 \cdot 10^{-15}, 0.475676159596792109 \cdot 10^{-14}$$

5

$$\text{tstData} := [a = 1.756, b = 4.76, c = 2.20, z = 2.85]$$

$$\text{TstData} := \frac{7908320945662591}{9007199254740992} \text{pow}(2, 1), \frac{2679641778285445}{4503599627370496} \text{pow}(2, 3), \frac{2476979795053773}{4503599627370496} \text{pow}(2, 2), \frac{6417629469002957}{9007199254740992} \text{pow}(2, 2)$$

$$\text{DLL} = 0.0231635195042665942 - 0.0188908693592305978 \text{ I}$$

$$\text{MPL} = 0.0231635195042665318 - 0.0188908693592298518 \text{ I}$$

$$\text{error}_{\text{absolute, relative}} = 0.748605209706691795 \cdot 10^{-15}, 0.250441299868204634 \cdot 10^{-13}$$

6

$$\text{tstData} := [a = 1.756, b = 0.556, c = 0.996, z = 0.0778162990325950434 - 0.000605574311537704596 \text{ I}]$$

$$\text{TstData} := \frac{7908320945662591}{9007199254740992} \text{pow}(2, 1), \frac{626000348204499}{1125899906842624} \text{pow}(2, 0), \frac{2242792614430507}{2251799813685248} \text{pow}(2, 0),$$

$$\frac{2803627642612369}{4503599627370496} \text{pow}(2, -3) - \frac{5585437171274447}{9007199254740992} \text{I pow}(2, -10)$$

$$\text{DLL} = 1.08323344382660181 - 0.000706748046851243469 \text{ I}$$

$$\text{MPL} = 1.08323344382660158 - 0.000706748046851243469 \text{ I}$$

$$\text{error}_{\text{absolute, relative}} = 0.23 \cdot 10^{-15}, 0.210000045692714432 \cdot 10^{-15}$$

7

$$\text{tstData} := [a = 1.756, b = 2.76, c = 2.20, z = 4]$$

$$\text{TstData} := \frac{7908320945662591}{9007199254740992} \text{pow}(2, 1), \frac{1553741871442821}{2251799813685248} \text{pow}(2, 2), \frac{2476979795053773}{4503599627370496} \text{pow}(2, 2), \frac{1}{2} \text{pow}(2, 3)$$

$$\text{DLL} = 0.0393278956028134327 - 0.0210256495322343750 \text{ I}$$

$$\text{MPL} = 0.0393278956028129817 - 0.0210256495322347949 \text{ I}$$

$$\text{error}_{\text{absolute, relative}} = 0.616211822346829045 \cdot 10^{-15}, 0.138180281342244844 \cdot 10^{-13}$$

8

$$\text{hypergeom}([1, 2], [3], z)$$

$$\text{tmp} := -\frac{2(\ln(1-z) + z)}{z^2}$$

$$\text{tstData} := [a = 1, b = 2.00000001, c = 3, z = 2]$$

$$\text{TstData} := \frac{1}{2} \text{pow}(2, 1), \frac{2251799824944247}{4503599627370496} \text{pow}(2, 2), \frac{3}{4} \text{pow}(2, 2), \frac{1}{2} \text{pow}(2, 2)$$

$$\text{DLL} = -1.00000001467399358 - 1.57079631108693518 \text{ I}$$

$$\text{MPL} = -1.00000001467401067 - 1.57079631108693341 \text{ I}$$

$$\text{error}_{\text{absolute, relative}} = 0.171814143771693022 \cdot 10^{-13}, 0.922695776098463467 \cdot 10^{-14}$$

9

$$\text{tstData} := [a = 1, b = 2., c = 3, z = 2]$$

$$\text{TstData} := \frac{1}{2} \text{pow}(2, 1), \frac{1}{2} \text{pow}(2, 2), \frac{3}{4} \text{pow}(2, 2), \frac{1}{2} \text{pow}(2, 2)$$

$$\text{DLL} = -0.99999999999881539 - 1.57079632679486569 \text{ I}$$

$$\text{MPL} = -1. - 1.57079632679489656 \text{ I}$$

$$\text{error}_{\text{absolute, relative}} = 0.122417177801973527 \cdot 10^{-12}, 0.657416951218880266 \cdot 10^{-13}$$

$$\text{tstData} := \left[a = 1, b = 2 - \frac{1}{384} \text{I}, c = 3, z = 4 \right]$$

$$\text{TstData} := \frac{1}{2} \text{pow}(2, 1), \frac{1}{2} \text{pow}(2, 2) - \frac{6004799503160661}{9007199254740992} \text{I pow}(2, -8), \frac{3}{4} \text{pow}(2, 2), \frac{1}{2} \text{pow}(2, 3)$$

$$\text{DLL} = -0.635184136492591644 - 0.392946040698066712 \text{ I}$$

$$\text{MPL} = -0.635184136492591311 - 0.392946040698071486 \text{ I}$$

$$\text{error}_{\text{absolute, relative}} = 0.478559975342694116 \cdot 10^{-14}, 0.640746853574339088 \cdot 10^{-14}$$

$$\text{tstData} := \left[a = 1.000000000000001, b = 2 - \frac{1}{384}I, c = 3, z = 4 \right]$$

$$\text{tstData} := [a = 1, b = 2, c = 3.1, z = 4]$$

$$\text{tstData} := [a = 1, b = 2.000000000000001 - 0.002604166666666666667I, c = 3.1, z = 4]$$

$$\text{tstData} := \left[a = 1, b = \frac{769}{384}, c = 3, z = 4 \right]$$

$$\text{TstData} := \frac{1}{2} \text{pow}(2, 1), \frac{4509463689385301}{9007199254740992} \text{pow}(2, 2), \frac{3}{4} \text{pow}(2, 2), \frac{1}{2} \text{pow}(2, 3)$$

$$\text{DLL} = -0.637073018206709474 - 0.390555746769017664I$$

$$\text{MPL} = -0.637073018206706365 - 0.390555746769017609I$$

$$\text{error}_{\text{absolute, relative}} = 0.310948645277640661 \cdot 10^{-14}, 0.416504941754002855 \cdot 10^{-14}$$

10

$$\text{tstData} := [a = 1.00000001, b = 2.00000002, c = -2.99999997, z = 2]$$

$$\text{TstData} := \frac{1125899918101623}{2251799813685248} \text{pow}(2, 1), \frac{1125899918101623}{2251799813685248} \text{pow}(2, 2), -\frac{3377699686750875}{4503599627370496} \text{pow}(2, 2), \frac{1}{2} \text{pow}(2, 2)$$

$$\text{DLL} = -0.106666669038266010 \cdot 10^{11} + 0.178813934326171875 \cdot 10^{-6}I$$

$$\text{MPL} = -0.106666669038249035 \cdot 10^{11} - 0.110214494010428357 \cdot 10^{-7}I$$

$$\text{error}_{\text{absolute, relative}} = 0.00169750001061486680, 0.159140000995147627 \cdot 10^{-12}$$

11

$$\text{tstData} := [a = 1, b = 2, c = 3, z = 2]$$

$$\text{TstData} := \frac{1}{2} \text{pow}(2, 1), \frac{1}{2} \text{pow}(2, 2), \frac{3}{4} \text{pow}(2, 2), \frac{1}{2} \text{pow}(2, 2)$$

$$\text{DLL} = -0.99999999999881539 - 1.57079632679486569I$$

$$\text{MPL} = -1. - 1.57079632679489656I$$

$$\text{error}_{\text{absolute, relative}} = 0.122417177801973527 \cdot 10^{-12}, 0.657416951218880266 \cdot 10^{-13}$$

$$\text{tstData} := \left[a = 1, b = 2 - \frac{1}{384}I, c = 3, z = 4 \right]$$

$$\text{TstData} := \frac{1}{2} \text{pow}(2, 1), \frac{1}{2} \text{pow}(2, 2) - \frac{6004799503160661}{9007199254740992}I \text{pow}(2, -8), \frac{3}{4} \text{pow}(2, 2), \frac{1}{2} \text{pow}(2, 3)$$

$$\text{DLL} = -0.635184136492591644 - 0.392946040698066712I$$

$$\text{MPL} = -0.635184136492591311 - 0.392946040698071486I$$

$$\text{error}_{\text{absolute, relative}} = 0.478559975342694116 \cdot 10^{-14}, 0.640746853574339088 \cdot 10^{-14}$$

12

$$\text{tstData} := [a = 1, b = 1.000001, c = 2.20, z = 4]$$

$$\text{tstData} := \left[a = 1, b = \frac{20000000001}{10000000000}, c = 3, z = 1 + \frac{1}{10000000000}I \right]$$

$$\text{TstData} := \frac{1}{2} \text{pow}(2, 1), \frac{2251799813696507}{4503599627370496} \text{pow}(2, 2), \frac{3}{4} \text{pow}(2, 2), \frac{1}{2} \text{pow}(2, 1) + \frac{6189700196426901}{9007199254740992}I \text{pow}(2, -36)$$

$$\text{DLL} = 48.6568720518369204 + 3.14159265336164628I$$

$$\text{MPL} = 48.6568720518358901 + 3.14159265336095617I$$

$$\text{error}_{\text{absolute, relative}} = 0.124006850701886628 \cdot 10^{-11}, 0.254340203770137345 \cdot 10^{-13}$$

13

$$\text{tstData} := \left[a = 1, b = \frac{200000001}{100000000}, c = 3, z = \frac{3}{2} \right]$$

$$\text{TstData} := \frac{1}{2} \text{pow}(2, 1), \frac{2251799824944247}{4503599627370496} \text{pow}(2, 2), \frac{3}{4} \text{pow}(2, 2), \frac{3}{4} \text{pow}(2, 1)$$

$$\text{DLL} = -0.717202540726473048 - 2.79252679462199316I$$

$$\text{MPL} = -0.717202540726473936 - 2.79252679462197984I$$

$$\text{error}_{\text{absolute, relative}} = 0.133495671839951425 \cdot 10^{-13}, 0.463199821924191814 \cdot 10^{-14}$$

14

15

$$\begin{aligned} \text{tstData} &:= \left[a = 4, b = 4.00000001, c = 5.00000001, z = \frac{3}{2} \right] \\ \text{tstData} &:= [a = 4., b = 4.00000001, c = 5.00000001, z = 1.5000000000000000] \\ \text{TstData} &:= \frac{1}{2} \text{pow}(2, 3), \frac{4503599638629495}{9007199254740992} \text{pow}(2, 3), \frac{5629499545472119}{9007199254740992} \text{pow}(2, 3), \frac{3}{4} \text{pow}(2, 1) \\ \text{DLL} &= -13.5847089402521650 + 2.48224608892963516 \text{ I} \\ \text{MPL} &= -13.5847089402520957 + 2.48224608892964183 \text{ I} \\ \text{error}_{\text{absolute, relative}} &= 0.696202477731873632 \cdot 10^{-13}, 0.504172876016462786 \cdot 10^{-14} \end{aligned}$$

16

$$\begin{aligned} \text{tstData} &:= \left[a = 1, b = 2, c = 3, z = \frac{9}{2} \right] \\ \text{TstData} &:= \frac{1}{2} \text{pow}(2, 1), \frac{1}{2} \text{pow}(2, 2), \frac{3}{4} \text{pow}(2, 2), \frac{9}{16} \text{pow}(2, 3) \\ \text{DLL} &= -0.568174120345239619 - 0.310280755910104789 \text{ I} \\ \text{MPL} &= -0.568174120345221523 - 0.310280755910103012 \text{ I} \\ \text{error}_{\text{absolute, relative}} &= 0.181830400373534898 \cdot 10^{-13}, 0.280885649612811124 \cdot 10^{-13} \\ \text{tstData} &:= \left[a = 1, b = 2 - \frac{1}{384} \text{ I}, c = 3, z = 1.9 \right] \\ \text{TstData} &:= \frac{1}{2} \text{pow}(2, 1), \frac{1}{2} \text{pow}(2, 2) - \frac{6004799503160661}{9007199254740992} \text{ I pow}(2, -8), \frac{3}{4} \text{pow}(2, 2), \frac{4278419646001971}{4503599627370496} \text{pow}(2, 1) \\ \text{DLL} &= -0.990214921748033605 - 1.73598085470367547 \text{ I} \\ \text{MPL} &= -0.990214921748102439 - 1.73598085470367369 \text{ I} \\ \text{error}_{\text{absolute, relative}} &= 0.688570109429678918 \cdot 10^{-13}, 0.344534942796702477 \cdot 10^{-13} \end{aligned}$$

17

$$\begin{aligned} \text{tstData} &:= [a = 1.756, b = 4.76, c = 2.20, z = 2.85] \\ \text{TstData} &:= \frac{7908320945662591}{9007199254740992} \text{pow}(2, 1), \frac{2679641778285445}{4503599627370496} \text{pow}(2, 3), \frac{2476979795053773}{4503599627370496} \text{pow}(2, 2), \frac{6417629469002957}{9007199254740992} \text{pow}(2, 2) \\ \text{DLL} &= 0.0231635195042665942 - 0.0188908693592305978 \text{ I} \\ \text{MPL} &= 0.0231635195042665318 - 0.0188908693592298518 \text{ I} \\ \text{error}_{\text{absolute, relative}} &= 0.748605209706691795 \cdot 10^{-15}, 0.250441299868204634 \cdot 10^{-13} \end{aligned}$$

tests: Forrey's tables

Forrey, table 2

$$\begin{aligned} &\text{hypergeom}\left(\left[\frac{1}{2}, 1\right], \left[\frac{3}{2}\right], -Z^2\right) \\ \text{exact} &:= \frac{-1}{2} \text{I} \ln\left(\frac{1-Z}{1+Z}\right) \\ \text{exact} \Big|_{Z=0.2} &= 0.986977799249403790 + 0.192312500000000000 \cdot 10^{-18} \text{ I} \\ Z &= 0.200000000000000000 \\ \text{exact} &= 0.986977799249403792 + 0.192312500000000000 \cdot 10^{-36} \text{ I} \\ \text{DLL} &= 0.986977799249403764 \\ \text{MPL} &= 0.986977799249403764 \\ \text{error}_{\text{absolute, relative}} &= 0., 0. \end{aligned}$$

$Z = 0.400000000000000000$
 $\text{exact} = 0.951265942780912216 + 0.301725000000000000 \cdot 10^{-36} \text{ I}$
 $\text{DLL} = 0.951265942780912144$
 $\text{MPL} = 0.951265942780912255$
 $\text{error}_{\text{absolute, relative}} = 0.111 \cdot 10^{-15}, 0.117 \cdot 10^{-15}$

$Z = 0.600000000000000000$
 $\text{exact} = 0.900699167117640259 - 0.392166666666666667 \cdot 10^{-36} \text{ I}$
 $\text{DLL} = 0.900699167117640309$
 $\text{MPL} = 0.900699167117640309$
 $\text{error}_{\text{absolute, relative}} = 0., 0.$

$Z = 0.800000000000000000$
 $\text{exact} = 0.843426177779440829 - 0.121937500000000000 \cdot 10^{-36} \text{ I}$
 $\text{DLL} = 0.843426177779440289$
 $\text{MPL} = 0.843426177779440844$
 $\text{error}_{\text{absolute, relative}} = 0.555 \cdot 10^{-15}, 0.658 \cdot 10^{-15}$

$Z = 1.000000000000000000$
 $\text{exact} = 0.785398163397448310 - 0. \text{ I}$
 $\text{DLL} = 0.785398163397448057$
 $\text{MPL} = 0.785398163397448279$
 $\text{error}_{\text{absolute, relative}} = 0.222 \cdot 10^{-15}, 0.283 \cdot 10^{-15}$

$Z = 1.200000000000000000$
 $\text{exact} = 0.730048375498494519 + 0.191258333333333333 \cdot 10^{-36} \text{ I}$
 $\text{DLL} = 0.730048375498494995$
 $\text{MPL} = 0.730048375498494551$
 $\text{error}_{\text{absolute, relative}} = 0.444 \cdot 10^{-15}, 0.61 \cdot 10^{-15}$

$Z = 1.400000000000000000$
 $\text{exact} = 0.678962029151482248 + 0.193035714285714286 \cdot 10^{-37} \text{ I}$
 $\text{DLL} = 0.678962029151482627$
 $\text{MPL} = 0.678962029151482294$
 $\text{error}_{\text{absolute, relative}} = 0.333 \cdot 10^{-15}, 0.49 \cdot 10^{-15}$

$Z = 1.600000000000000000$
 $\text{exact} = 0.632623132157083865 - 0.737343750000000000 \cdot 10^{-37} \text{ I}$
 $\text{DLL} = 0.632623132157083545$
 $\text{MPL} = 0.632623132157083878$
 $\text{error}_{\text{absolute, relative}} = 0.333 \cdot 10^{-15}, 0.526 \cdot 10^{-15}$

$Z = 1.800000000000000000$
 $\text{exact} = 0.590943234668088701 - 0.733750000000000000 \cdot 10^{-37} \text{ I}$
 $\text{DLL} = 0.590943234668088402$
 $\text{MPL} = 0.590943234668088735$
 $\text{error}_{\text{absolute, relative}} = 0.333 \cdot 10^{-15}, 0.564 \cdot 10^{-15}$

$Z = 2.000000000000000000$
 $\text{exact} = 0.553574358897045252 - 0. \text{ I}$
 $\text{DLL} = 0.553574358897044760$
 $\text{MPL} = 0.553574358897045204$

$$\text{error}_{\text{absolute, relative}} = 0.444 \cdot 10^{-15}, 0.802 \cdot 10^{-15}$$

$$Z = 2.2000000000000000$$

$$\text{exact} = 0.520076742576372968 + 0.9340000000000000 \cdot 10^{-37} I$$

$$\text{DLL} = 0.520076742576372597$$

$$\text{MPL} = 0.520076742576372930$$

$$\text{error}_{\text{absolute, relative}} = 0.333 \cdot 10^{-15}, 0.640 \cdot 10^{-15}$$

 Forrey, table 2, modified

$$\text{hypergeom}\left(\left[\frac{1}{2}, 1\right], \left[\frac{3}{2}\right], Z\right)$$

$$\text{exact} := \frac{1}{2} \frac{\ln\left(-\frac{1+\sqrt{Z}}{-1+\sqrt{Z}}\right)}{\sqrt{Z}}$$

$$z = 0.2$$

$$\text{exact} = 1.07602235241001010$$

$$\text{DLL} = 1.07602235241001010$$

$$\text{MPL} = 1.07602235241001010$$

$$\text{error}_{\text{absolute, relative}} = 0., 0.$$

$$z = 0.4000000000000000$$

$$\text{exact} = 1.17873607983194814$$

$$\text{DLL} = 1.17873607983194750$$

$$\text{MPL} = 1.17873607983194817$$

$$\text{error}_{\text{absolute, relative}} = 0.67 \cdot 10^{-15}, 0.568 \cdot 10^{-15}$$

$$z = 0.6000000000000000$$

$$\text{exact} = 1.33194290062992538$$

$$\text{DLL} = 1.33194290062992438$$

$$\text{MPL} = 1.33194290062992526$$

$$\text{error}_{\text{absolute, relative}} = 0.88 \cdot 10^{-15}, 0.661 \cdot 10^{-15}$$

$$z = 0.8000000000000000$$

$$\text{exact} = 1.61403352861501515$$

$$\text{DLL} = 1.61403352861501448$$

$$\text{MPL} = 1.61403352861501515$$

$$\text{error}_{\text{absolute, relative}} = 0.67 \cdot 10^{-15}, 0.415 \cdot 10^{-15}$$

$$z = 1.0000000000000000$$

$$\text{exact} = \text{Float}(\infty) + 1.57079632679489662 I$$

$$\text{DLL} = \text{Float}(\text{undefined}) + \text{Float}(\text{undefined}) I$$

$$\text{MPL} = \text{Float}(\text{undefined}) + \text{Float}(\text{undefined}) I$$

$$\text{error}_{\text{absolute, relative}} = \text{Float}(\text{undefined}), \text{Float}(\text{undefined})$$

$$z = 1.2$$

$$\text{exact} = 1.40991541361497965 + 1.43393430238636911 I$$

$$\text{DLL} = 1.40991541361497941 - 1.43393430238636843 I$$

$$\text{MPL} = 1.40991541361497985 - 1.43393430238636910 I$$

$$\text{error}_{\text{absolute, relative}} = 0.801560977094069868 \cdot 10^{-15}, 0.398621064736827202 \cdot 10^{-15}$$

$$z = 1.4000000000000000$$

$$\text{exact} = 1.04709910157357316 + 1.32756519890263223 I$$

$$\text{DLL} = 1.04709910157357311 - 1.32756519890263158 I$$

$$\text{MPL} = 1.04709910157357333 - 1.32756519890263225 I$$

$$\text{error}_{\text{absolute, relative}} = 0.705195008490559298 \cdot 10^{-15}, 0.417349724679223759 \cdot 10^{-15}$$

$$z = 1.6000000000000000$$

$$\text{exact} = 0.848240113644489548 + 1.24182353322451265 \text{ I}$$

$$\text{DLL} = 0.848240113644489147 - 1.24182353322451200 \text{ I}$$

$$\text{MPL} = 0.848240113644489480 - 1.24182353322451267 \text{ I}$$

$$\text{error}_{\text{absolute, relative}} = 0.748190483767335817 \cdot 10^{-15}, 0.497728309083229955 \cdot 10^{-15}$$

$$z = 1.8000000000000000$$

$$\text{exact} = 0.717348234940006731 + 1.17080245517345440 \text{ I}$$

$$\text{DLL} = 0.717348234940006102 - 1.17080245517345372 \text{ I}$$

$$\text{MPL} = 0.717348234940006657 - 1.17080245517345438 \text{ I}$$

$$\text{error}_{\text{absolute, relative}} = 0.862336941108288599 \cdot 10^{-15}, 0.628004436002374002 \cdot 10^{-15}$$

$$z = 2.0000000000000000$$

$$\text{exact} = 0.623225240140230513 + 1.11072073453959156 \text{ I}$$

$$\text{DLL} = 0.623225240140230285 - 1.11072073453959130 \text{ I}$$

$$\text{MPL} = 0.623225240140230508 - 1.11072073453959153 \text{ I}$$

$$\text{error}_{\text{absolute, relative}} = 0.320357612676833388 \cdot 10^{-15}, 0.251370712644621033 \cdot 10^{-15}$$

$$z = 2.2$$

$$\text{exact} = 0.551767364575178460 + 1.05903066748288517 \text{ I}$$

$$\text{DLL} = 0.551767364575178276 - 1.05903066748288466 \text{ I}$$

$$\text{MPL} = 0.551767364575178387 - 1.05903066748288510 \text{ I}$$

$$\text{error}_{\text{absolute, relative}} = 0.453785191472793656 \cdot 10^{-15}, 0.380278420193566343 \cdot 10^{-15}$$

$$z = 2.4000000000000000$$

$$\text{exact} = 0.495445490177582727 + 1.01394466899340297 \text{ I}$$

$$\text{DLL} = 0.495445490177582659 - 1.01394466899340263 \text{ I}$$

$$\text{MPL} = 0.495445490177582770 - 1.01394466899340308 \text{ I}$$

$$\text{error}_{\text{absolute, relative}} = 0.463487863918786983 \cdot 10^{-15}, 0.410263281496045588 \cdot 10^{-15}$$

Forrey, table 3

$$\text{hypergeom}([1, 1], [2], z)$$

$$\text{exact} := -\frac{\ln(1-z)}{z}$$

$$\text{exact} \Big|_{z=0.1} = 1.05360515657826301$$

$$z = 0.1$$

$$\text{exact} = 1.05360515657826301$$

$$\text{DLL} = 1.05360515657826292$$

$$\text{MPL} = 1.05360515657826292$$

$$\text{error}_{\text{absolute, relative}} = 0., 0.$$

$$z = 0.2000000000000000$$

$$\text{exact} = 1.11571775657104878$$

$$\text{DLL} = 1.11571775657104855$$

$$\text{MPL} = 1.11571775657104877$$

$$\text{error}_{\text{absolute, relative}} = 0.22 \cdot 10^{-15}, 0.197 \cdot 10^{-15}$$

$$z = 0.3000000000000000$$

$$\text{exact} = 1.18891647979577460$$

$$\text{DLL} = 1.18891647979577475$$

$$\text{MPL} = 1.18891647979577453$$

error_{absolute, relative} = 0.22 10⁻¹⁵, 0.19 10⁻¹⁵

z = 0.40000000000000000000
exact = 1.27706405941497671
DLL = 1.27706405941497647
MPL = 1.27706405941497669

error_{absolute, relative} = 0.22 10⁻¹⁵, 0.172 10⁻¹⁵

z = 0.50000000000000000000
exact = 1.38629436111989062
DLL = 1.38629436111989013
MPL = 1.38629436111989057

error_{absolute, relative} = 0.44 10⁻¹⁵, 0.317 10⁻¹⁵

z = 0.60000000000000000000
exact = 1.52715121979025844
DLL = 1.52715121979025925
MPL = 1.52715121979025836

error_{absolute, relative} = 0.89 10⁻¹⁵, 0.58 10⁻¹⁵

z = 0.70000000000000000000
exact = 1.71996114903705142
DLL = 1.71996114903705077
MPL = 1.71996114903705122

error_{absolute, relative} = 0.45 10⁻¹⁵, 0.262 10⁻¹⁵

z = 0.80000000000000000000
exact = 2.01179739054262547
DLL = 2.01179739054262408
MPL = 2.01179739054262585

error_{absolute, relative} = 0.177 10⁻¹⁴, 0.880 10⁻¹⁵

z = 0.90000000000000000000
exact = 2.55842788110449520
DLL = 2.55842788110449471
MPL = 2.55842788110449559

error_{absolute, relative} = 0.88 10⁻¹⁵, 0.344 10⁻¹⁵

z = 1.00000000000000000000
exact = Float(∞)

Error, (in hypergeom) numeric exception: division by zero

Forrey, table 4

hypergeom([5], [], -Z)

$$\text{exact} := \frac{1}{(1+Z)^5}$$

$$\text{exact}|_{Z=-2.4} = -0.185934432081870649$$

$$Z = -2.4$$

$$\text{exact} = -0.185934432081870649$$

$$\text{DLL} = -0.185934432081870826 - 0.113852003550172017 \cdot 10^{-15} \text{ I}$$

$$\text{MPL} = -0.185934432081870715$$

$$\text{error}_{\text{absolute, relative}} = 0.159007165600762727 \cdot 10^{-15}, 0.857286384859496540 \cdot 10^{-15}$$

Z = -2.2000000000000000
exact = -0.401877572016460905
DLL = -0.401877572016460793 - 0.246079041109534315 10⁻¹⁵ I
MPL = -0.401877572016460627
error_{absolute, relative} = 0.296834793232511541 10⁻¹⁵, 0.736912440976175078 10⁻¹⁵

Z = -2.0000000000000000
exact = -1.0000000000000000
DLL = -1. - 0.612323399573676628 10⁻¹⁵ I
MPL = -1.
error_{absolute, relative} = 0.612323399573676628 10⁻¹⁵, 0.612323399573676628 10⁻¹⁵

Z = -1.8000000000000000
exact = -3.0517578125000000
DLL = -3.05175781249999956 - 0.186866271842552650 10⁻¹⁴ I
MPL = -3.05175781249999911
error_{absolute, relative} = 0.192208229668593497 10⁻¹⁴, 0.630428382661714033 10⁻¹⁵

Z = -1.6000000000000000
exact = -12.8600823045267490
DLL = -12.8600823045267383 - 0.787452931550509335 10⁻¹⁴ I
MPL = -12.8600823045267401
error_{absolute, relative} = 0.807763653185442072 10⁻¹⁴, 0.628124148290339389 10⁻¹⁵

Z = -1.4
exact = -97.6562500000000000
DLL = -97.6562500000001137 - 0.597972069896169238 10⁻¹³ I
MPL = -97.6562500000001137
error_{absolute, relative} = 0.597972069896169238 10⁻¹³, 0.612323399573676588 10⁻¹⁵

Z = -1.2000000000000000
exact = -3125.0000000000000000
DLL = -3124.9999999999955 - 0.191351062366773914 10⁻¹¹ I
MPL = -3125.000000000000364
error_{absolute, relative} = 0.451548700661283046 10⁻¹¹, 0.144604285747880358 10⁻¹⁴

Z = -1.0000000000000000
exact = Float(∞)
DLL = Float(undefined) + Float(undefined) I
MPL = Float(∞) + Float(∞) I
error_{absolute, relative} = Float(∞), Float(undefined)

Z = -0.8000000000000000
exact = 3125.0000000000000000
DLL = 3125.000000000000364
MPL = 3125.000000000000364
error_{absolute, relative} = 0., 0.

Z = -0.6000000000000000
exact = 97.6562500000000000
DLL = 97.6562499999999716
MPL = 97.6562499999999716
error_{absolute, relative} = 0., 0.

Z = -0.4
 exact = 12.8600823045267490
 DLL = 12.8600823045267507
 MPL = 12.8600823045267507
 error absolute, relative = 0., 0.

Z = -0.20000000000000000
 exact = 3.0517578125000000
 DLL = 3.05175781249999911
 MPL = 3.0517578125000000
 error absolute, relative = $0.89 \cdot 10^{-15}$, $0.292 \cdot 10^{-15}$

Z = 0.
 exact = 1.
 DLL = 1.
 MPL = 1.
 error absolute, relative = 0., 0.

Z = 0.20000000000000000
 exact = 0.401877572016460905
 DLL = 0.401877572016460960
 MPL = 0.401877572016460904
 error absolute, relative = $0.56 \cdot 10^{-16}$, $0.14 \cdot 10^{-15}$

Z = 0.40000000000000000
 exact = 0.185934432081870649
 DLL = 0.185934432081870715
 MPL = 0.185934432081870632
 error absolute, relative = $0.83 \cdot 10^{-16}$, $0.45 \cdot 10^{-15}$

Z = 0.6
 exact = 0.0953674316406250000
 DLL = 0.0953674316406249722
 MPL = 0.0953674316406250000
 error absolute, relative = $0.278 \cdot 10^{-16}$, $0.292 \cdot 10^{-15}$

Z = 0.80000000000000000
 exact = 0.0529221494013446460
 DLL = 0.0529221494013446397
 MPL = 0.0529221494013446397
 error absolute, relative = 0., 0.

Z = 1.00000000000000000
 exact = 0.03125000000000000
 DLL = 0.03125000000000000
 MPL = 0.03125000000000000
 error absolute, relative = 0., 0.

Z = 1.20000000000000000
 exact = 0.0194037913455985992
 DLL = 0.0194037913455985915
 MPL = 0.0194037913455986019
 error absolute, relative = $0.104 \cdot 10^{-16}$, $0.536 \cdot 10^{-15}$

Z = 1.40000000000000000

exact = 0.0125586741255144033
 DLL = 0.0125586741255144050
 MPL = 0.0125586741255144050
 error_{absolute, relative} = 0., 0.

Z = 1.6
 exact = 0.00841653357321576221
 DLL = 0.00841653357321575993
 MPL = 0.00841653357321575993
 error_{absolute, relative} = 0., 0.

Forrey, table 5

$$\text{exact} := \frac{\text{hypergeom}([1, 2 + 10^{(-i)}], [3], 3)}{2 \left(4 \cdot 100^i \cdot (-2)^{(-2 - 10^{(-i)})} - 100^i - 3 \cdot 10^i \right)}$$

$$\text{exact} \Big|_{i=1} = -\frac{100 \cdot (-1)^{(9/10)} \cdot 2^{(9/10)}}{99} - \frac{260}{99}$$

$$= -0.83360180501122916 - 0.582470809489636332 \text{ I}$$

z = 3, b = 2.1000000000000000, i = 1
 exact = -0.833601805011229157 - 0.582470809489636333 I
 DLL = -0.833601805011223940 - 0.582470809489636476 I
 MPL = -0.833601805011229158 - 0.582470809489636254 I
 error_{absolute, relative} = 0.522272036394827480 10⁻¹⁴, 0.513572688840767045 10⁻¹⁴

z = 3, b = 2.0100000000000000, i = 2
 exact = -0.822828069542514518 - 0.686331993761715218 I
 DLL = -0.822828069542484997 - 0.686331993761715498 I
 MPL = -0.822828069542514529 - 0.686331993761715498 I
 error_{absolute, relative} = 0.29532 10⁻¹³, 0.275613560036705790 10⁻¹³

z = 3, b = 2.0010000000000000, i = 3
 exact = -0.820920943130888842 - 0.696949863001329951 I
 DLL = -0.820920943130840408 - 0.696949863001330328 I
 MPL = -0.820920943130888814 - 0.696949863001330105 I
 error_{absolute, relative} = 0.484065136629359004 10⁻¹³, 0.449512688412626743 10⁻¹³

z = 3, b = 2.0001000000000000, i = 4
 exact = -0.820721617721787828 - 0.698013498839889593 I
 DLL = -0.820721617721784424 - 0.698013498839889612 I
 MPL = -0.820721617721787866 - 0.698013498839889390 I
 error_{absolute, relative} = 0.344915177978586498 10⁻¹⁴, 0.320127635958564074 10⁻¹⁴

z = 3, b = 2.0000100000000000, i = 5
 exact = -0.820701598757428930 - 0.698119880420660619 I
 DLL = -0.820701598757426320 - 0.698119880420659533 I
 MPL = -0.820701598757428985 - 0.698119880420660532 I
 error_{absolute, relative} = 0.284608959802744088 10⁻¹⁴, 0.264172152091575531 10⁻¹⁴

z = 3, b = 2.0000010000000000, i = 6
 exact = -0.820699595996464860 - 0.698130518758212335 I
 DLL = -0.820699595996459141 - 0.698130518758211638 I
 MPL = -0.820699595996464915 - 0.698130518758212193 I
 error_{absolute, relative} = 0.580061212287117160 10⁻¹⁴, 0.538392139139012527 10⁻¹⁴

z = 3, b = 2.0000001000000000, i = 7
exact = -0.820699395711722877 - 0.698131582593761758 I
DLL = -0.820699395711720792 - 0.698131582593762290 I
MPL = -0.820699395711722790 - 0.698131582593761957 I
error absolute, relative = 0.202555992258930716 10⁻¹⁴, 0.187980193844183315 10⁻¹⁴

z = 3, b = 2.0000001000000000, i = 8
exact = -0.820699375683162223 - 0.698131688977334642 I
DLL = -0.820699375683167420 - 0.698131688977333575 I
MPL = -0.820699375683162202 - 0.698131688977334686 I
error absolute, relative = 0.533496438601046333 10⁻¹⁴, 0.495099476050807274 10⁻¹⁴

z = 3, b = 2.0000000100000000, i = 9
exact = -0.820699373680305293 - 0.698131699615692110 I
DLL = -0.820699373680331190 - 0.698131699615687773 I
MPL = -0.820699373680305322 - 0.698131699615691992 I
error absolute, relative = 0.262097955924879353 10⁻¹³, 0.243255452703243272 10⁻¹³

z = 3, b = 2.0000000010000000, i = 10
exact = -0.820699373480019591 - 0.698131700679527859 I
DLL = -0.820699373480031413 - 0.698131700679528899 I
MPL = -0.820699373480019645 - 0.698131700679527900 I
error absolute, relative = 0.118103270488162181 10⁻¹³, 0.109612788201390207 10⁻¹³

z = 3, b = 2.0000000001000000, i = 11
exact = -0.820699373459991021 - 0.698131700785911434 I
DLL = -0.820699373459999881 - 0.698131700785910470 I
MPL = -0.820699373459991000 - 0.698131700785911469 I
error absolute, relative = 0.893701079780034729 10⁻¹⁴, 0.829618185318182307 10⁻¹⁴

z = 3, b = 2.0000000000100000, i = 12
exact = -0.820699373457988164 - 0.698131700796549791 I
DLL = -0.820699373457998149 - 0.698131700796551513 I
MPL = -0.820699373457988157 - 0.698131700796549737 I
error absolute, relative = 0.101486077862926598 10⁻¹³, 0.941753199301120840 10⁻¹⁴

z = 3, b = 2.0000000000010000, i = 13
exact = -0.820699373457787878 - 0.698131700797613627 I
DLL = -0.820699373457790982 - 0.698131700797611776 I
MPL = -0.820699373457787873 - 0.698131700797613775 I
error absolute, relative = 0.369619831719024514 10⁻¹⁴, 0.343169035424973684 10⁻¹⁴

z = 3, b = 2.0000000000001000, i = 14
exact = -0.820699373457767850 - 0.698131700797720010 I
DLL = -0.820699373457774883 - 0.698131700797719912 I
MPL = -0.820699373457767889 - 0.698131700797719801 I
error absolute, relative = 0.699488077096386254 10⁻¹⁴, 0.649118376707608215 10⁻¹⁴

z = 3, b = 2.0000000000000100, i = 15
exact = -0.820699373457765847 - 0.698131700797730649 I
DLL = -0.820699373457773329 - 0.698131700797732901 I
MPL = -0.820699373457765780 - 0.698131700797730792 I
error absolute, relative = 0.783806621559170040 10⁻¹⁴, 0.727011413295163767 10⁻¹⁴

```

z = 3, b = 2.0000000000000010, i = 16
exact = -0.820699373457765647 - 0.698131700797731713 I
DLL = -0.820699373457816184 - 0.698131700797724464 I
MPL = -0.820699373457765668 - 0.698131700797731791 I
error_absolute, relative = 0.510445999592513214 10-13, 0.473778397331663995 10-13

```

more tests: bad results

And finally some examples, where all that does not behave so nice

```

tstData := [a = 15.7, b = 2.98, c = 16.7, z = 0.911 e(1 / 4 I π)]
TstData :=  $\frac{4419157134357299}{4503599627370496}$  pow(2, 4),  $\frac{6710363444782039}{9007199254740992}$  pow(2, 2),  $\frac{4700632111067955}{9007199254740992}$  pow(2, 5),
 $\frac{5802206073670979}{9007199254740992}$  pow(2, 0) +  $\frac{5802206073670979}{9007199254740992}$  I pow(2, 0)
DLL = -2.51202952622992015 + 0.476029721509771409 I
MPL = -2.51202952101332455 + 0.476029726166660283 I
error_absolute, relative = 0.699281657407974258 10-8, 0.273505666449399810 10-8

```

This is due to cancellation errors by the linear transforms, where numbers of almost the same magnitude are subtracted: the 'heads' cancel out, but for the desired 'tails' - needed for the actual result - the computing precision is not good enough.

Also note, that the parameters have to be of quite moderate magnitude (the 'asymptotic' case of larger parameters is not implemented, I would have to read deeper stuff written by Nico Temme for that).

```

tstData := [a = 60.7, b = 2.98, c = 15.7, z = 0.911 e(1 / 4 I π)]
TstData :=  $\frac{4271382771584205}{4503599627370496}$  pow(2, 6),  $\frac{6710363444782039}{9007199254740992}$  pow(2, 2),  $\frac{4419157134357299}{4503599627370496}$  pow(2, 4),
 $\frac{5802206073670979}{9007199254740992}$  pow(2, 0) +  $\frac{5802206073670979}{9007199254740992}$  I pow(2, 0)
DLL = -4.55496629897718996 - 17.5770728908178526 I
MPL = 0.0269620168628056239 + 0.00540291571183267646 I
error_absolute, relative = 18.1696869147131439, 660.763136136640163

```

testing through the ODE for hypergeometric functions

An 'intrinsic' test would be: do the results satisfy the hypergeometric differential equation?

$$ode := b a f(z) + ((1 + a + b) z - c) \left(\frac{d}{dz} f(z) \right) + (-z + z^2) \left(\frac{d^2}{dz^2} f(z) \right) = 0$$

$$b a \text{hypergeom}([a, b], [c], z) + \frac{((1 + a + b) z - c) b a \text{hypergeom}([1 + b, a + 1], [c + 1], z)}{c}$$

$$+ \frac{(-z + z^2) b a (1 + b) (a + 1) \text{hypergeom}([2 + b, a + 2], [c + 2], z)}{c (c + 1)} = 0$$

$$ODE := b a H(a, b, c, z) + \frac{((1 + a + b) z - c) b a H(1 + b, a + 1, c + 1, z)}{c} + \frac{z(z - 1) b a (1 + b) (a + 1) H(2 + b, a + 2, c + 2, z)}{c (c + 1)}$$

```

tstData := [a = 1.756, b = 4.76, c = 2.20, z = 1.95 I]
check := eval(ODE, TST)

```

$$check|_{H=MPL_2F1} = -0.1 10^{-17} + 0. I$$

$$check|_{H=hyp2f1_DLL} = 0.1149 10^{-14} + 0.1810 10^{-14} I$$

```

DLL = -0.0291190062043904692 - 0.0104896762749490478 I
MPL = -0.0291190062043906114 - 0.0104896762749490859 I
error_absolute, relative = 0.147215658134588387 10-15, 0.475676159596792109 10-14

```

```
tstData := [ a = 1.00000001, b = 2.00000002, c = -2.99999997, z = 2 ]
check := eval(ODE, TST)
```

```
check|H = MPL_2F1 = -29.866669 + 0.281478409394186851 10-5 I
```

```
check|H = hyp2f1_DLL = 0.6 10-5 - 0.512599987983704854 10-5 I
```

But that cross check is of limited use: here it asserts, that the DLL is not fine, but better than Maple (giving a smaller error) - which certainly is not the case ... but here one would have to work with much better precision, even if using Maple only:

```
oldDigits := 18
```

```
Digits := 36
```

```
check|H = MPL_2F1 = -0.59733338 10-16 + 0.562956815121596261735 10-23 I
```

On the other hand: for that test data the relative error is not so large, it is ~ 2E-13 (but has a sign problem for the imaginary part):

```
DLL = -0.106666669038266010 1011 + 0.178813934326171875 10-6 I
```

```
MPL = -0.106666669038249035 1011 - 0.110214494010428357 10-7 I
```

```
errorabsolute, relative = 0.00169750001061486680, 0.159140000995147627 10-12
```